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THE PROBLEM OF EMBEDDING S^n INTO R^{n+1} WITH PRESCRIBED GAUSS CURVATURE AND ITS SOLUTION BY VARIATIONAL METHODS

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ABSTRACT. A way to recover a closed convex hypersurface from its Gauss curvature is to find a positive function over S^n whose graph would represent the hypersurface in question. Then one is led to a nonlinear elliptic problem of Monge-Ampère type on S^n. Usually, geometric problems involving operators of this type are too complicated to be suggestive for a natural functional whose critical points are candidates for solutions of such problems. It turns out that for the problem indicated in the title, such a functional exists and has interesting geometric properties. With the use of this functional, we obtain new existence results for hypersurfaces with prescribed curvature as well as strengthen some that are already known.

Introduction. In his book on convex polyhedrons Aleksandrov posed a general question of finding variational formulations and solutions to several geometric problems related to convex bodies [**A**, Chapter 7, §1, section 4]. As far as we know, until now such a solution for closed convex hypersurfaces is known only for the celebrated Minkowski problem.

In this paper we present a variational solution to the following problem: Under what restrictions can a positive function $K(X)$, $X \in R^{n+1}$, $n \geq 2$, be realized as the Gauss-Kronecker curvature of some closed convex hypersurface in R^{n+1}? See Yau [**Y**, p. 683].

In [**O**], and also [**D**], it was shown that if $K \in C^k(R^{n+1})$, $k \geq 3$, and some other conditions are satisfied then the hypersurface in question can be recovered as a graph of a smooth positive function ρ over a unit sphere S^n in R^{n+1}. The function ρ must satisfy on S^n a Monge-Ampère type equation of the form

$$(*) \qquad (\rho^2 + |\nabla\rho|^2)^{-n/2-1}\rho^{-2n+2}\frac{\det(-\rho\,\mathrm{Hess}\,\rho + 2\nabla\rho \times \nabla\rho + \rho^2 e)}{\det(e)} = K,$$

where $\nabla\rho$ and $\mathrm{Hess}\,\rho$ denote correspondingly the gradient and Hessian of ρ in the standard metric e on S^n, and K is evaluated at the point $X = (x, \rho(x))$, $x \in S^n$.

Our main purpose in this paper is to construct and investigate a variational problem for which equation $(*)$ is the Euler-Lagrange equation. The approach to solving $(*)$ via variational calculus allows construction of solutions in the geometrically natural class of closed convex hypersurfaces not subject to any smoothness restrictions. The only "smoothness" requirement on the data is that the given function is continuous. The natural question when the solution to the variational problem is smooth will be treated in a separate publication.

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It should be pointed out that in the known cases when an equation of Monge-Ampère type is the Euler-Lagrange equation for some functional, one can demonstrate this connection (at least formally) by explicit computation of the variation combined with a straightforward integration by parts. The form of the functional is usually quite clear from the geometry of the problem. This is not the case with equation (*). Though a posteriori the constructed functional seems geometrically natural, if one attempts to differentiate it to produce (*), one quickly realizes that the step involving integration by parts is a formidable task (except for the case $n = 1$) because of the complicated form of the equation. However, by geometric means this can be done, provided that (*) is reinterpreted to make sense for nonsmooth convex hypersurfaces.

A few historical remarks are perhaps in order here. In general, only a few functionals are known for which concrete equations of Monge-Ampère type are the Euler-Lagrange equations. For the Minkowski problem such a functional arises naturally as the volume integral, and Minkowski used the variational formulation to find a weak solution to his problem [**M**]. Later Hilbert [**H**], Aleksandrov [**A1**], and Fenchel and Jessen [**FJ**] generalized this approach and strengthened the results of Minkowski. For hypersurfaces that are graphs of convex (or concave) functions over bounded domains in R^n, Bakelman [**B, B1**] investigated a functional analogous to the volume integral and showed existence and uniqueness of the minimum. The functional considered by Bakelman is obtained by integration by parts from the functional given in [**CH**, p. 326]. Finally, in the Weyl problem of isometric embedding in R^3 of S^2 with a metric of positive curvature a variational formulation was suggested by Blaschke and Herglotz in [**BH**]. The solution of the corresponding variational problem took a long time and was carried out by Volkov [**V**].

The paper is organized as follows. In §1 we define weak solutions of equation (*), introduce the functional associated with (*), and point out some simple properties of it. In §2 we show that this functional is differentiable in the class of convex polyhedrons, and we compute its derivative. In §3 we give conditions for the variational problem to admit a solution in class of convex polyhedrons. A condition for uniqueness is also given here. Finally, in §4, using Minkowski's method of approximating an arbitrary convex hypersurface by convex polyhedrons, we give conditions for existence of weak solutions of (*). Also, we prove here for arbitrary convex hypersurfaces the differentiabilty result established for polyhedrons in §2.

1. The main functional.

1.1 Throughout the paper the term *closed convex hypersurface* refers to the boundary of a compact convex body in R^{n+1} with interior points.

Fix a Cartesian coordinate system in R^{n+1} with origin at some point $\mathbf{O}$. Let F be a closed convex hypersurface starshaped relative to $\mathbf{O}$. We parametrize F by the *radial* map $r\colon S^n \to F$, where S^n is a unit sphere centered at $\mathbf{O}$, and $r(x) = \rho(x)x$, $x \in S^n$, $\rho(x)$ is the distance from $\mathbf{O}$ to the point of intersection with F of the ray originating at $\mathbf{O}$ in the direction x. (Here and elsewhere in the paper we identify a point x on S^n with the corresponding unit vector originating at $\mathbf{O}$.) The function $\rho(x)$ below is called the *distance* function of F.

Since F is convex, at every point of it there exists at least one supporting hyperplane. We assume that all supporting hyperplanes to F are oriented so that their normals are directed in the half-spaces containing F.

Let $x \in S^n$ and $X = r(x) \in F$. The (*generalized*) *Gauss map* γ assigns to a point $X \in F$ and a supporting hyperplane p at X a point $u \in S^n$ such that the vector u is of the same direction as the normal to p (see [**Bu**]).

Consider the composite map $\alpha_F = \gamma \circ r: S^n \to S^n$. Let $\omega \subseteq S^n$ be a Borel set. Then $r(\omega)$ is of Borel type on F, and it is well known that $\gamma(r(\omega))$ is again a Borel set on S^n [**Bu**]. Hence, α_F induces an automorphism of the σ-algebra of S^n.

1.1.1 Suppose now that, in addition to being a closed convex hypersurface, F is of class C^2 and $\rho(x) > 0$ on S^n. Then for the Gauss curvature K we have an expression (see, for example, [**O**])

$$K = (\rho^2 + |\nabla\rho|^2)^{-(n+1)/2}\rho^{-2n+2}\frac{\det(-\rho \operatorname{Hess}\rho + 2\nabla\rho \otimes \nabla\rho + \rho^2 e)}{\det(e)}.$$

It will be useful to write down the formula for the unit normal vector field on F:

$$u = (\nabla\rho - \rho x)/\sqrt{\rho^2 + |\nabla\rho|^2}.$$

The expression $\rho^n K$ splits naturally into two parts:

$$\alpha_F^* = (\rho^2 + |\nabla\rho|^2)^{-(n+1)/2}\rho^{-n+1}\frac{\det(-\rho \operatorname{Hess}\rho + 2\nabla\rho \times \nabla\rho + \rho^2 e)}{\det(e)},$$

which is the Jacobian of the map α_F, and

$$\langle x, u\rangle = -\rho/\sqrt{\rho^2 + |\nabla\rho|^2},$$

where $\langle\ ,\ \rangle$ denotes the standard inner product in R^{n+1}.

For any Borel set $\omega \subset S^n$ we have

$$\int_\omega \rho^n K\, d\sigma = -\int_\omega \langle x, u\rangle \alpha_F^*\, d\sigma = -\int_{\alpha_F(\omega)} \langle x, u\rangle \mu(F, d\sigma),$$

where $\mu(F, d\sigma)$ is the n-dimensional volume element of S^n evaluated on the Gauss image of $r(\omega)$. The last integral on the right makes sense, however, for nonsmooth convex hypersurfaces, and this is the basis for everything that follows. Namely, for a given continuous function $\phi(X) \geq 0$, $X \in R^{n+1}$, we write X as (x, ρ), $x \in S^n$, $\rho = |X|$, and call a closed convex hypersurface F with distance function $\rho(x)$ a *weak solution* of the equation $\rho^n K = \phi(x, \rho)$ if for any Borel subset ω of S^n the relation

$$\int_\omega \phi(x, \rho(x))\, d\sigma = -\int_{\alpha_F(\omega)} \langle x, u\rangle \mu(F, d\sigma)$$

is satisfied.

This approach of defining nonsmooth convex solutions has been utilized by Minkowski [**M**], Aleksandrov [**A**], Pogorelov [**P, P1**], Bakelman [**B**], and others.

1.2 Now, as in section 1.1, let F be a closed convex hypersurface (not necessarily smooth) starshaped relative to the origin **O** and $\rho(x)$, $x \in S^n$, its distance function. On F $-\langle x, u\rangle > 0$ and the quantity

$$\mathbf{J}(F, \omega) = -\int_{\alpha_F(\omega)} \langle x, u\rangle \mu(F, d\sigma)$$

determines a finite nonnegative function on Borel subsets of S^n. It is completely additive as it follows from complete additivity of $\mu(F, d\sigma)$ (see [**A2**, Chapter 5]) and boundedness of $|\langle x, u\rangle|$. Hence, we may consider the functional

$$\mathbf{J}(\rho) = \int_{S^n} \rho(x)\mathbf{J}(F, d\omega). \tag{1.1}$$

Observe that if F is a sphere of radius $\rho = c = \text{const}$ then $\mathbf{J}(\rho) = c\sigma_n$, where σ_n is the n-dimensional volume of S^n.

1.3 The domain of definition of $\mathbf{J}(\rho)$ can be extended to the class of all nonnegative continuous functions on S^n. Let $v(x)$ be such a function; that is, $v(x) \in C(S^n)$, $v(x) \geq 0$, and $v(x) \not\equiv 0$. Denote by V the closed hypersurface defined by the map $r(x) = v(x)x\colon S^n \to R^{n+1}$, and let H be the boundary of the convex hull of the set of points $\{(x, v(x)) | x \in S^n\}$. We put

$$\mathbf{J}(v) = \int_{S^n} v\mathbf{J}(H, d\omega).$$

It is not difficult to see that if $\rho(x)$ is the distance function of H then $\mathbf{J}(v) = \mathbf{J}(\rho)$. This observation will be used later several times.

1.4 It is of particular importance to us to consider the function $\mathbf{J}$ on closed convex polyhedrons. Let $x_1, x_2, \dots, x_m$, $m \geq 3$, be points on S^n not all lying in one closed hemisphere. Denote by $l_1, l_2, \dots, l_m$ the rays originating at $\mathbf{O}$ in the directions $x_1, x_2, \dots, x_m$, and let $X_1, X_2, \dots, X_m$ be points lying on corresponding rays at positive distances $\rho_1, \rho_2, \dots, \rho_m$. The boundary of the convex hull of the set $\{X_1, X_2, \dots, X_m\}$ is a closed convex polyhedron starshaped relative to $\mathbf{O}$. Denote it by P. Of course, not all of the points $X_1, X_2, \dots, X_m$ may lie on P. But vertices of P may occur only on the rays $l_1, l_2, \dots, l_m$. We do not exclude here the case when some of the points X_i are not "true" vertices on P; that is, they are allowed to be interior points of faces of P of dimension ≥ 1. We denote by $\mathbf{P}$ the class of all closed convex polyhedrons with only possible vertices lying on this fixed set of rays.

Let $P \in \mathbf{P}$, and let $\rho(x)$ be the distance function of P. Then P is uniquely determined by the values $\rho_i = \rho(x_i)$. The functional $\mathbf{J}$ for P reduces to the function

$$\mathbf{J}(\rho) = -\sum_{i=1}^{m} \rho_i \left\langle x_i, \int_{\alpha_P(x_i)} u\mu(P, d\sigma) \right\rangle.$$

If T is a closed polyhedron (not necessarily convex) with only possible vertices on the rays $l_1, l_2, \dots, l_m$, $v(x)$ its distance function, and $v_i = v(x_i)$, then the extended function $\mathbf{J}$ is given by

$$\mathbf{J}(v) = -\sum_{i=1}^{m} v_i \left\langle x_i, \int_{\alpha_H(x_i)} u\mu(H, d\sigma) \right\rangle,$$

where H is the boundary of the convex hull of the set $\{x_i, v(x_i)\}$, $i = 1, 2, \dots, m$.

2. The first variation of the functional J on polyhedrons.

2.1 Let $P \in \mathbf{P}$ be a closed convex polyhedron with positive distance function $\rho(x)$, and $h = (h_1, h_2, \dots, h_m)$ an arbitray vector in R^m. Denote by P' the polyhedron obtained from P by displacing the vertices of P along the rays $l_1, l_2, \dots, l_m$ correspondingly by the distances $h_1, h_2, \dots, h_m$. If the length $|h|$ is sufficiently small, P' will be starshaped relative to $\mathbf{O}$, though not necessarily convex. Denote by $\rho_n(x)$ the distance function of P' and consider the function $\mathbf{J}(\rho_h)$ defined as at the end of 1.4.

2.2 THEOREM. *For the closed covnex polyhedron P the first variation of the function $\mathbf{J}$ is defined and is the principal part in h in the expression*

$$\mathbf{J}(\rho_h) - \mathbf{J}(\rho) = -\sum_{i=1}^{m} h_i \left\langle x_i, \int_{\alpha_P(x_i)} u\mu(P, d\sigma) \right\rangle + o(|h|).$$

2.2.1 PROOF. Consider first the case when the vector $h = (0, \ldots, 0, h_i, 0, \ldots, 0)$ for some fixed i.

On the polyhedron P let $X_i = (x_i, \rho_i)$. Two possibilities may occur: (a) X_i is a true vertex; that is,

$$\mu_i = \int_{\alpha_P(x_i)} \mu(P, d\sigma) > 0,$$

and (b) X_i is an interior point of a face of dimension k on P where $1 \le k \le n$. We start with (a). The Gauss image $\gamma(x_i)$ on S^n is a convex (on S^n) n-dimensional set whose boundary consists of pieces of $(n-1)$-dimensional great[2] spheres. The same is true with respect to all other vertices.

Clearly, each side of $\gamma(X_i)$ is formed by the set of normals to supporting hyperplanes to P containing an edge joining X_i with some adjacent vertex X_j. Note that the sides of $\gamma(X_i)$ lie in hyperplanes perpendicular to the corresponding edges X_iX_j.

The polyhedron P' is obtained from P by displacing the vertex X_i along the ray l_i by the distance h_i. Let H be the convex polyhedron bounding the convex hull of the points $(x_1, \rho_1), \ldots, (x_{i-1}, \rho_{i-1}), (x_i, \rho_i + h_i), (x_{i+1}, \rho_{i+1}), \ldots, (x_m, \rho_m)$, and $v(x)$ the distance function of H. Put $v_j = v(x_j)$, $j = 1, 2, \ldots, m$.

We have

$$\begin{aligned}\mathbf{J}(\rho_h) - \mathbf{J}(\rho) = &-h_i \left\langle x_i, \int_{\alpha_H(x_i)} u\mu(H, d\sigma) \right\rangle \\ &+ \rho_i \left\langle x_i, \int_{\alpha_P(x_i)} u\mu(P, d\sigma) - \int_{\alpha_H(x_i)} u\mu(H, d\sigma) \right\rangle \\ &+ \sum_{j \ne i} \rho_j \left\langle x_j, \int_{\alpha_P(x_j)} u\mu(P, d\sigma) - \int_{\alpha_H(x_j)} u\mu(H, d\sigma) \right\rangle.\end{aligned}$$

Now note that for any closed convex polyhedron P,

$$\sum_{j=1}^{m} \int_{\alpha_P(x_j)} u\mu(p, d\sigma) = 0,$$

and, in particular, we have the identity

$$\begin{aligned}&\int_{\alpha_P(x_i)} u\mu(P, d\sigma) - \int_{\alpha_H(x_i)} u\mu(H, d\sigma) \\ &\qquad = -\sum_{j \ne i} \left[\int_{\alpha_P(x_j)} u\mu(P, d\sigma) - \int_{\alpha_H(x_j)} u\mu(H, d\sigma) \right].\end{aligned}$$

[2]A great sphere is the intersection of S^n with a hyperplane in R^{n+1} passing through the center of S^n.

Consider now the expression

$$\sum_{j\neq i}\left\langle \rho_j x_j - \rho_i x_i, \int_{\alpha_P(x_j)} u\mu(P,d\sigma) - \int_{\alpha_H(x_j)} u\mu(H,d\sigma)\right\rangle.$$

An analysis of the set $\alpha_H(x_j)$ on S^n shows that either $\alpha_H(x_j) = \alpha_P(x_j)$ or, at most,

$$\left\langle \rho_j x_j - \rho_i x_i, \int_{\alpha_P(x_j)} u\mu(P,d\sigma) - \int_{\alpha_H(x_j)} u\mu(H,d\sigma)\right\rangle = o(|h|),$$

because the edge X_iX_j is perpendicular to the normals u_{ij} of supporting hyperplanes to P containing this edge, and on the set $\alpha_P(x_j)\backslash\alpha_H(x_j)$ $u = u_{ij} + a$ for some vectors u_{ij} and a, where $|a| = O(|h|)$.

Then, taking into account that

$$\int_{\alpha_H(x_i)} u\mu(H,d\sigma) = \int_{\alpha_P(x_i)} u\mu(P,d\sigma) + O(|h|),$$

we conclude that

$$\mathbf{J}(\rho_h) - \mathbf{J}(\rho) = -h_i\left\langle x_i, \int_{\alpha_P(x_i)} u\mu(P,d\sigma)\right\rangle + o(|h|).$$

If case (b) occurs, that is, when $\mu_i = 0$, the same arguments apply; the only difference is that in this case $\mathbf{J}(\rho_h) - \mathbf{J}(\rho) = o(|h|)$. Finally, the general case of an arbitrary vector h (of sufficiently small length) is obtained by considering a superposition of individual displacements of each of the vertices of P. The theorem is proved.

2.2.2 REMARK. It is clear from the proof of the above theorem that one does not need to assume that $\rho(x_i) > 0$ for all $i = 1, 2, \ldots, m$. One can allow $\rho(x_i) = 0$ for some i, provided the variation vector h is such that for the polyhedron P' $\rho(x_i) + h_i > 0$ for all sufficiently small h.

3. Existence and uniqueness results for polyhedrons.

3.1 For any polyhedron P from the class $\mathbf{P}$ introduce the quantities

$$\mathbf{J}_i(P) = \mathbf{J}(P, x_i) = -\left\langle x_i, \int_{\alpha_P(x_i)} u\mu(P,d\sigma)\right\rangle, \qquad i = 1, 2, \ldots, m.$$

Thus, $\mathbf{J}(\rho) = \sum_{i=1}^m \rho_i \mathbf{J}_i(P)$. $\mathbf{J}_i(P)$ represents the n-dimensional area of the projection of $\alpha_P(x_i)$ on the hyperplane tangent to S^n at the point x_i.

3.2 THEOREM. *Put $\mathbf{R}^+ = (0,\infty)$. Let $\phi_i(t)\colon \mathbf{R}^+ \to [0,\infty)$, $i = 1, 2, \ldots, m$, be continuous functions, $\underline{P}$, $\overline{P}$ two closed convex polyhedrons from the class $\mathbf{P}$ and $\underline{\rho}(x)$, $\overline{\rho}(x)$ their corresponding distance functions. Assume that $0 < \underline{\rho}_i \le \overline{\rho}_i$, $i = 1, 2, \ldots, m$, where $\underline{\rho}_i = \underline{\rho}(x_i)$, $\overline{\rho}_i = \overline{\rho}(x_i)$, and*

$$\phi_i(\underline{\rho}_i) \ge \mathbf{J}_i(\underline{P}), \tag{3.1}$$

$$\phi_i(\overline{\rho}_i) \le \mathbf{J}_i(\overline{P}), \qquad i = 1, 2, \ldots, m. \tag{3.2}$$

Further let

$$\mathbf{P}_s = \{P \in \mathbf{P} | \underline{\rho}_i \leq \rho_i \leq \overline{\rho}_i, \ i = 1, 2, \dots, m\},$$

where $\rho_i = \rho(x_i)$.

Then the function

$$Q(\rho) \equiv \mathbf{J}(\rho) - \sum_{i=1}^{m} \int_{\varepsilon}^{\rho_i} \phi_i(t)\, dt,$$

where $0 < \varepsilon < \min_{S^n} \underline{\rho}(x)$, *attains on* $\mathbf{P}_s$ *its absolute minimum on some polyhedron* $P^0 \in \mathbf{P}_s$ *which either coincides with* $\underline{P}$ *or* $\overline{P}$, *or its distance function* $\rho^0(x)$ *satisfies the inequality* $\underline{\rho}(x) < \rho^0(x) < \overline{\rho}(x)$, $x \in S^n$.

3.2.1 PROOF. Put $\Omega = \times_{i=1}^{m} [\underline{\rho}_i, \overline{\rho}_i]$. For any vector $(\rho_1, \rho_2, \dots, \rho_m) \in \Omega$ we may consider the convex hull of the points $\rho_i x_i$, $i = 1, 2, \dots, m$, and put $\mathbf{J}(\rho_1, \rho_2, \dots, \rho_m) = \sum_{i=1}^{m} \rho_i \mathbf{J}_i(P)$, where P is the boundary of this convex hull. Thus, the function $Q(\rho_1, \rho_2, \dots, \rho_m)$ is defined on Ω. Further, if $v(x)$ is the distance function of P, then $\sum_{i=1}^{m} \rho_i \mathbf{J}_i(P) = \sum_{i=1}^{m} v(x_i) \mathbf{J}_i(P)$, and it follows from Theorem 2.2 that $Q(\rho_1, \rho_2, \dots, \rho_m)$ is continuous on Ω.

Moreover, since $Q(\rho_1, \rho_2, \dots, \rho_m) \geq Q(v(x_1), v(x_2), \dots, v(x_m))$, we may assume that the minimum of Q is attained on a vector $(\rho_1^0, \rho_2^0, \dots, \rho_m^0) \in \Omega$ corresponding to a closed convex polyhedron P^0 from $\mathbf{P}_s$; that is, $\rho_i^0 x_i \in P^0$.

Suppose that for some j $\rho_j^0 = \overline{\rho}_j$. Then, evidently, $\mathbf{J}_j(P^0) \geq \mathbf{J}_j(\overline{P})$.

If $\mathbf{J}_j(P^0) = \mathbf{J}_j(\overline{P})$ then the polyhedrons $\overline{P}$ and P^0 coincide along the faces for which $X_j = (x_j, \rho_j^0)$ is a boundary point. Move to the vertex X_k adjacent to X_j. For this vertex $\rho_k^0 = \overline{\rho}_k$, and therefore we may again compare $\mathbf{J}_k(P^0)$ and $\mathbf{J}_k(\overline{P})$. Eventually, we will either arrive at a situation where $P^0 = \overline{P}$, or there will be found a vertex X_t such that $\mathbf{J}_t(P^0) > \mathbf{J}_t(\overline{P})$ and $\rho_t^0 = \overline{\rho}_t$. Only the latter case needs to be considered.

Clearly, $\mathbf{J}_t(P^0) > 0$. Hence, there exists a positive h, small enough so that by displacing the vertex X_t of P^0 by the distance h towards the origin and taking the boundary of the convex hull of the points $\{(x_i, \rho_i), i = 1, 2, \dots, t-1, t+1, \dots, m, (x_t, \rho_t^0 - h)\}$ we obtain a convex polyhedron P^1 in class $\mathbf{P}_s$. Let $\rho^0(x)$ be the distance function of P^0 and $\rho_h(x)$ the distance function of P^1. Applying Theorem 2.2 we get

$$Q(\rho^0) - Q(\rho_h) = h[J_t(P^0) - \phi_t(\overline{\rho}_t)] + o(h).$$

In view of (3.2) we now conclude that $Q(\rho^0) > Q(\rho_h)$, which contradicts the fact that $Q(\rho^0) = \inf Q(\rho)$ on P_S.

Now consider the situation when at X_t $\underline{\rho}_t = \rho_t^0 = \overline{\rho}_t$. However, this case does not occur for the following reason. Obviously, $\underline{\rho}(x) \leq \rho^0(x) \leq \overline{\rho}(x)$. At the point X_t we have $\mathbf{J}_t(\underline{P}) \geq \mathbf{J}_t(P^0) > \mathbf{J}_t(\overline{P})$. This and inequalities (3.1), (3.2) imply that $\phi_t(\underline{\rho}_t) > \phi_t(\underline{\rho}_t)$—a contradiction!

Finally, we note that the case where $\rho_j^0 = \underline{\rho}_j$ for some j is treated similarly. The theorem is proved.

3.2.2 REMARK. In the proof of Theorem 3.2 we needed to consider deformations of P^0 such that $\underline{\rho}_i + h > 0$. The assumption that ρ_i, $i = 1, 2, \dots, m$, are strictly positive was not used. Therefore, if the functions $\phi_i(t)$, $i = 1, 2, \dots, m$, are defined also for $t = 0$, so that the function $Q(\rho)$ makes sense, then it is sufficient to assume only that $0 \leq \underline{\rho}_i \leq \overline{\rho}_i$, and $\underline{P}$ does not degenerate into $\mathbf{O}$.

3.3 COROLLARY. *Suppose all conditions of the Theorem 3.2 are satisfied. Then there exists a convex polyhedron $P^0 \in \mathbf{P}_s$ such that $\mathbf{J}_i(P^0) = \phi_i(\rho_i^0)$, $i = 1, 2, \ldots, m$.*

3.3.1 PROOF. If $\phi_i(\rho_i) = \mathbf{J}_i(\underline{P})$ or $\phi_i(\overline{\rho}_i) = \mathbf{J}_i(\overline{P})$ for all $i = 1, 2, \ldots, m$, the statement is trivial. Thus, we can assume that at least for one index i $\phi_i(\underline{\rho}_i) > \mathbf{J}_i(\underline{P})$ and at least for one index j $\phi_j(\overline{\rho}_j) < \mathbf{J}_j(\overline{P})$. Consider again the function $Q(\rho)$ and the polyhedron P^0 on which $Q(\rho)$ attains its absolute minimum on $\mathbf{P}_s$. Suppose $P^0 = \overline{P}$. Make a homothetic transformation of P^0 with coefficient $0 < \lambda < 1$ relative to the origin $\mathbf{O}$. Assuming that λ is sufficiently close to 1, and considering the polyhedron with distance function $\rho_h(x) = \rho^0(x)(1 - \lambda)$, we get

$$Q(\rho^0) - Q(\rho_h) = (1 - \lambda) \sum_{i=1}^{m} [J_i(P^0) - \phi_i((\overline{\rho}_i))] + o(1 - \lambda).$$

Hence, $Q(\rho^0) > Q(\rho_h)$, and we arrived at a contradiction. Thus, $P^0 \neq \overline{P}$. Similarly, one shows that $P^0 \neq \underline{P}$. Then by Theorem 3.2 the distance function $\rho^0(x)$ of P^0 satisfies the inequalities $\underline{\rho}(x) < \rho^0(x) < \overline{\rho}(x)$ for all $x \in S^n$. Under such circumstances we may apply Theorem 2.2 and conclude that

$$0 = \left. \frac{\partial Q(\rho)}{\partial \rho_i} \right|_{\rho = \rho^0(x)} = J_i(P^0) - \phi_i(\rho_i^0), \qquad i = 1, 2, \ldots, m.$$

The corollary is proved.

3.3.2 REMARK. Suppose all conditions of Theorem 3.2 are satisfied except for (3.1) and (3.2), which are replaced by

$$\phi_i(\underline{\rho}_i) = J_i(\underline{P}), \quad \phi_i(\overline{\rho}_i) = J_i(\overline{P}), \qquad i = 1, 2, \ldots, m.$$

It is of interest to find out under what additional conditions (if any) on ϕ_i, $i = 1, 2, \ldots, m$, there exists a polyhedron $P^0 \neq \underline{P}$, $\overline{P}$ and such that

$$\phi_i(\rho_i^0) = \mathbf{J}_i(P^0), \qquad i = 1, 2, \ldots, m.$$

3.4 THEOREM. *Suppose the conditions of Theorem 3.2 are satisfied and, in addition, the functions $\phi_i(t)$, $i = 1, 2, \ldots, m$, are nonincreasing. Then the polyhedron P^0, whose existence is asserted in the corollary, is unique up to a homothetic transformation relative to $\mathbf{O}$.*

3.4.1 PROOF. Let P^0 and P^1 be two closed convex polyhedrons from $\mathbf{P}$ such that $\mathbf{J}_i(P^k) = \phi_i(\rho_i^k)$, $i = 1, 2, \ldots, m$, $k = 0, 1$. The polyhedrons P^0 and P^1 either intersect each other, or one of them, say P^1, lies inside the convex body bounded by P^0. In either case we make a homothetic transformation with coefficient λ of P^0 so that the transformed polyhedron λP^0 lies entirely inside the convex body bounded by P^1 and touches it at least at one point. Obviously, $\lambda \leq 1$.

Note that $\mathbf{J}_j(P^k)$, $i = 1, 2, \ldots, m$, are invariant under homothetic transformations. It is clear that, unless $\lambda P^0 = P^1$, there exists a vertex $(x_j, \rho_j^1) = (x_j, \lambda \rho_j^0)$ such that $\mathbf{J}_j(P^0) > \mathbf{J}_j(P^1)$. Then we have

$$\phi_j(\rho_j^0) = \mathbf{J}_j(P^0) > \mathbf{J}_j(P^1) = \phi_j(\rho_j^1) = \phi_j(\lambda \rho_j^0) \geq \phi(\rho_j^0),$$

which is a contradiction.

3.4.2 REMARK. For smooth (at least C^2) convex hypersurfaces the above theorem is essentially contained in [**A3**], and we use here the same idea for the proof.

Similarly one can prove that if P^0 and P^1 are two closed convex polyhedrons from class $\mathbf{P}$, then either P^0 is homothetic to P^1 or there exist a direction x_j for which $\mathbf{J}_j(P^0) > \mathbf{J}_j(P^1)$ and a direction x_t for which $\mathbf{J}_t(P^0) < \mathbf{J}_t(P^1)$.

4. Existence results for arbitrary closed convex hypersurfaces. It is well known that an arbitrary closed convex hypersurface can be approximated by closed convex polyhedrons. It is also known that the integral Gauss curvatures of the polyhedrons converge weakly to the integral Gauss curvature of the limiting hypersurface [**A2**, Chapter 5, **Bu**]. We need to establish a similar result for the measures

$$\mathbf{J}(F,\omega) = -\int_{\alpha_F(\omega)} \langle x, u\rangle \mu(F, d\omega), \qquad \omega \in B(S^n),$$

where $B(S^n)$ is the σ-algebra of Borel subsets of S^n.

4.1 THEOREM. *Let F_m be a sequence of closed convex hypersurfaces starshaped relative to $\mathbf{O}$ and converging to a closed convex hypersurface F which is also starshaped relative to $\mathbf{O}$. Then $\mathbf{J}(F_m, \omega)$ converge weakly to $\mathbf{J}(F, \omega)$, $\omega \in B(S^n)$.*

4.1.1 REMARK. The distance between F_m and F is understood in the usual sense; that is,

$$\operatorname{dist}(F_m, F) = \max\left\{\sup_{X\in F} \operatorname{dist}(X, F_m),\ \sup_{X\in F_m} \operatorname{dist}(X, F)\right\}.$$

4.1.2 PROOF OF THEOREM 4.1. One needs to show that for any continuous function f on S^n

$$\lim_{m\to\infty} \int_{S^n} f(x)\mathbf{J}(F_m, d\omega) = \int_{S^n} f(x)\mathbf{J}(F, d\omega).$$

For convenience consider another copy $\tilde{S}^n$ of the unit n-sphere in R^{n+1} centered at a point $\tilde{\mathbf{O}}$ different from $\mathbf{O}$ and assume that the Gauss map γ of F_m and F is a map into $\tilde{S}^n$ instead of S^n, leaving S^n to play the role of the manifold over which F and F_m are defined. Then $\alpha_F, \alpha_{F_m}\colon S^n \to \tilde{S}^n$.

Let

$$U = \{u \in \tilde{S}^n | \alpha_F^{-1}(u) \text{ consists of more than one point on } S^n\}.$$

It is known that $\sigma(U) = 0$, where σ is the standard n-dimensional measure on $\tilde{S}^n$ [**A2**, Chapter V]. Therefore, almost everywhere on $\tilde{S}^n$ the function $x(u) = \alpha_F^{-1}(u)$ is well defined, and

$$\mathbf{J}(F,\omega) = -\int_{\alpha_F(\omega)} \langle x(u), u\rangle\, d\sigma.$$

Similarly, the functions $x_m(u)$ are defined almost everywhere on $\tilde{S}^n$ for the hypersurfaces F_m, and

$$\mathbf{J}(F_m,\omega) = -\int_{\alpha_{F_m}(\omega)} \langle x_m(u), u\rangle\, d\sigma.$$

Then

$$\int_{S^n} f(x)\mathbf{J}(F, d\omega) = -\int_{\tilde{S}^n} f(x(u))\langle x(u), u\rangle\, d\sigma, \tag{4.1}$$

$$\int_{S^n} f(x)\mathbf{J}(F_m, d\omega) = -\int_{\tilde{S}^n} f(x_m(u))\langle x_m(u), u\rangle\, d\sigma. \tag{4.2}$$

Since the sequence F_m converges to F, the position vectors of F_m (as functions of u) converge everywhere where they are defined to the position vector of F (also as a function of u), and consequently $x_m(u)$ converge to $x(u)$ almost everywhere on $\tilde{S}^n$. Under such circumstances, applying Lebesgue's theorem, we conclude that

$$\lim_{m\to\infty}\int_{\tilde{S}^n} f(x_m(u))\langle x_m(u), u\rangle\, d\sigma = \int_{\tilde{S}^n} f(x(u))\langle x(u), u\rangle\, d\sigma.$$

This, together with (4.1), (4.2), completes the proof of the theorem.

4.2 Now we extend formula (2.1) for the derivatives of $\mathbf{J}(\rho)$ to arbitrary closed convex hypersurfaces.

THEOREM. *Let F be a closed convex hypersurface starshaped relative to the origin $\mathbf{O}$. Suppose its distance function $\rho(x)$ is positive on S^n, and let $h(x)$ be an arbitrary continuous function on S^n such that $\rho_h(x) = \rho(x) + h(x) \geq 0$. Then the functional $\mathbf{J}$ defined by* (1.1) *admits the first variation, and it is the principal part in the expression*

$$\mathbf{J}(\rho_n) - \mathbf{J}(\rho) = \int_{S^n} h\mathbf{J}(F, d\omega) + o(\|h\|),$$

where $\|h\|$ is the maximum norm in $C(S^n)$.

4.2.1 PROOF. The theorem is proved by approximating F by convex polyhedrons. Choose on S^n points $x_1, x_2, \ldots, x_m$ so that the convex hull of the points $(x_i, \rho(x_i))$, $i = 1, 2, \ldots, m$, contains the origin $\mathbf{O}$ inside. Let P_m be the boundary of this convex hull and $\rho^m(x)$ its distance function.

There exists a natural partition of S^n into closed subdomains V_i such that $\operatorname{diam} V_i \leq \varepsilon_m$, $i = 1, 2, \ldots, m$, where ε_m is a sequence of positive numbers converging to zero. Namely, put

$$W_i = \{x \in S^n | \alpha_{P_m}(x) \subset \alpha_{P_m}(x_i)\}, \qquad i = 1, 2, \ldots, m.$$

If $\operatorname{diam} W_i \leq \varepsilon_m$, then put $V_i = W_i$. If $\operatorname{diam} W_i > \varepsilon_m$, then subdivide W_i into V_{i_k} so that $\bigcup_k V_{i_k} = W_i$, $\operatorname{diam} V_{i_k} \leq \varepsilon_m$ for all k, and x_k is an interior point of V_{i_k}. Then put $V_k = V_{i_k}$. Clearly, $\bigcup_{i=1}^m V_i = S^n$ and x_i is an interior point of V_i for all i. It is known (see [**A2**, Chapter 5]) that weak convergence of the measures $\mathbf{J}(P_m, \omega)$ implies that $\lim_{m\to\infty} \mathbf{J}(\rho^m) = \mathbf{J}(\rho)$, since ρ^m are continuous on S^n and converge uniformly (in $C(S^n)$) to ρ.

Now we turn to the hypersurface F_h defined by the distance function $\rho_h(x) = \rho(x) + h(x)$. Denote by $\tilde{F}_h$ the boundary of the convex hull of F_h. By definition

$$\mathbf{J}(\rho_h) = \int_{S^n} \rho_h(x)\mathbf{J}(\tilde{F}_h, d\omega) = \int_{S^n} \tilde{\rho}_h(x)\mathbf{J}(\tilde{F}_h, d\omega),$$

where $\tilde{\rho}_h(x)$ is the distance function of $\tilde{F}_h$.

Denote by P'_m the convex polyhedron defined by vertices $(x_i, \tilde{\rho}_h(x_i))$, $i = 1, 2, \ldots, m$, and let ρ'_m be its distance function. By Theorem 2.2

$$\begin{aligned}\mathbf{J}(\rho'_m) - \mathbf{J}(\rho_m) &= -\sum_{i=1}^m h(x_i)\left\langle x_i, \int_{\alpha_{P_m}(x_i)} u\mu(P_m, d\sigma)\right\rangle + o\left(\left|\max_{1\leq i\leq m} h(x_i)\right|\right) \\ &= \int_{S^n} h'_m \mathbf{J}(P_m, d\omega) + o(\|h'\|),\end{aligned}$$

where $h'(x) = \rho'_m(x) - \rho_m(x)$, $x \in S^n$, and $\|h'\| = \max_{S^n} |h'|$. Applying once again Aleksandrov's theorem [**A2**, Chapter 2], we can pass to the limit when $\varepsilon_m \to 0$ and conclude that

$$\mathbf{J}(\rho_h) - \mathbf{J}(\rho) = \int_{S^n} h\mathbf{J}(F, d\omega) + o(\|h\|).$$

4.3 THEOREM. *Let $\phi(x,t)\colon S^n \times R^+ \to [0,\infty)$ be a continuous function and R_1, R_2 two numbers such that $0 < R_1 \le R_2 < \infty$. Assume that*

$$\phi(x, R_1) \ge 1, \qquad x \in S^n, \tag{4.3}$$

$$\phi(x, R_2) \le 1, \qquad x \in S^n. \tag{4.4}$$

Then there exists a closed convex hypersurface F located between two concentric spheres of radii R_1 and R_2 with centers at the origin **O** *and such that for any Borel subset $\omega \subset S^n$*

$$\mathbf{J}(F, \omega) = \int_\omega \phi(x, \rho(x))\, d\sigma,$$

where $\rho(x)$ is the distance function of F.

4.3.1 PROOF. On S^n choose points $x_1, x_2, \dots, x_m$ so that the convex hull of this set of points contains the origin **O** inside. Let $\hat{P}_m$ be the boundary of this convex hull. Define for $i = 1, 2, \dots, m$

$$V_i = \{x \in S^n | \alpha_{\hat{P}_m}(x) \subset \alpha_{\hat{P}_m}(x_i)\}.$$

Clearly, $\bigcup_{i=1}^m V_i = S^n$ and x_i is an interior point of V_i. Assume that the points $x_1, x_2, \dots, x_m$ are chosen so that $\operatorname{diam} V_i < \varepsilon_m$, where ε_m is a small positive number such that the preceding requirements to $\hat{P}_m$ are satisfied.

Let

$$\tilde{\phi}_i(t) = \int_{V_i} \langle x_i, x\rangle \phi(x,t)\, d\sigma, \qquad t \in [0, \infty),\ i = 1, 2, \dots, m.$$

The functions $\tilde{\phi}_1(t), \tilde{\phi}_2(t), \dots, \tilde{\phi}_m(t)$ satisfy the conditions of Theorem 3.2. Indeed, for a fixed i we have, in view of (4.3),

$$\tilde{\phi}_i(R_1) \ge \int_{V_i} \langle x_i, x\rangle\, d\sigma = -\int_{\alpha_{P_m}(x_i)} \langle x_i, u\rangle \mu(\hat{P}_m, d\sigma) = J_i(\hat{P}_m).$$

Thus, the polyhedron $R_1\hat{P}_m$, obtained from $\hat{P}_m$ by a homothetic transformation with coefficient R_1 relative to the origin **O** can be taken as the polyhedron $\underline{P}$ in Theorem 3.2. We also use here the fact that $\mathbf{J}_i(\hat{P}_m) = \mathbf{J}_i(R_1\hat{P}_m)$.

Similarly, $\rho_i(R_2) \le \mathbf{J}_i(P_m) = \mathbf{J}_i(R_2\hat{P}_m)$. Under such circumstances it follows from Corollary 3.3 that there exists a convex polyhedron P_m such that $\mathbf{J}_i(P_m) = \phi_i(\rho_i^m)$, $i = 1, 2, \dots, m$, where $\rho^m(x)$ is the distance function of P_m and $\rho_i^m = \rho^m(x_i)$. In addition, $R_1 \le \rho_i^m \le R_2$ for all i.

Now we let m tend to infinity so that $\varepsilon_m \to 0$. The corresponding set of polyhedrons $\{P_m\}$ contains a converging subsequence according to Blaschke's compactness theorem. We denote this subsequence again by $\{P_m\}$; its limit is a closed convex hypersurface F with distance function $\rho(x)$. Clearly, $R_1 \le \rho(x) \le R_2$, $x \in S^n$.

Let ω be any Borel set on S^n. For any polyhedron P_m we have

$$\mathbf{J}(P_m, \omega) = \sum \mathbf{J}_i(P_m) = \sum \int_{V_i} \langle x_i, x\rangle \phi(x, \rho^m(x_i))\, d\sigma,$$

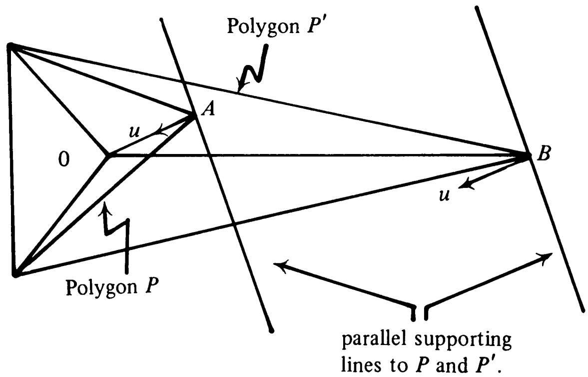

FIGURE 1

where the summation is over all vertices x_i of $\hat{P}_m$ that belong to ω. By a standard argument from real analysis one shows that

$$\lim_{m\to\infty}\sum\int_{V_i}\langle x_i, x\rangle\phi(x,\rho^m(x_i))\,d\sigma = \int_\omega \phi(x,\rho(x))\,d\sigma.$$

On the other hand, by Theorem 4.2 $\mathbf{J}(P_m,\omega)$ converge weakly to $\mathbf{J}(F,\omega)$. Since the weak limit is unique, we conclude that for any Borel set $\omega\subset S^n$,

$$\mathbf{J}(F,\omega) = \int_\omega \phi(x,\rho(x))\,d\sigma,$$

and the theorem is proved.

5. Concluding remarks.

5.1 Assumptions (4.3), (4.4) in Theorem 4.3 as well as their version (3.1), (3.2) in Theorem 3.2 cannot, in general, be omitted. This can be seen from the following argument. A simple analysis shows that on any closed convex hypersurface F starshaped relative to the origin $\mathbf{O}$ the relation

$$2\,\mathrm{Volume}(B_n) < \mathbf{J}(F,S^n) \le \sigma_n$$

holds; here B_n denotes the n-dimensional ball of radius 1. Hence, if, for example, the function $\phi(X) > 1$ for all $X\in R^{n+1}$, then there is no solution to the problem

$$\mathbf{J}(F,\omega) = \int_\omega \phi(x,\rho(x))\,d\sigma,$$

where ω is a Borel set on S^n.

5.2 The question of uniqueness in Theorem 4.3 remains open even if it is assumed that $\phi(X)$ is a monotone function of $|X|$. For smooth convex hypersurfaces this assumption is sufficient, and essentially uniqueness is a consequence of Aleksandrov's uniqueness theorem (see [**O**]).

5.3 In the same framework as in 1.1.1 one can define weak solutions to the equation

$$\int_\omega f(x)\,d\sigma = \int_{\alpha_F(\omega)} \nu(X,u)\mu(F,d\sigma), \qquad \omega\in S^n,$$

where f is a given nonnegative function on S^n, F a closed convex hypersurface to be found, and $\nu(X,u)$ a positive continuous function on $R^{n+1}\times S^n$. This was done by

Pogorelov [**P1**, Chapter 8] under the following monotonicity assumption on $\nu(X,u)$: for any $u \in S^n$ $\nu(X,u) \leq \nu(X',u)$ whenever $|X| < |X'|$. Figure 1 shows that in the case studied in this paper the function $-\langle x,u\rangle$ does not satisfy this monotonicity condition. In this example $-\langle \mathbf{O}A/|\mathbf{O}A|,u\rangle = 1 > -\langle \mathbf{O}B/|\mathbf{O}B|,u\rangle$. The methods of Pogorelov for investigation of the above problem are different from ours and rely crucially on the monotonicity condition. On the other hand, a functional similar to (1.1) can also be defined in the case studied by Pogorelov. However, the formulas for variation given in Theorems 2.2 and 4.2 do not seem to extend for arbitrary functions $\nu(X,u)$.

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